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AMONG the various ways of conceiving man's affection for life there is one which perhaps Metchnikoff has not heretofore investigated, yet in this one way that desire has a majestic aspect. It is quite different from the way one usually regards the feeling of fear of death. There come moments when the mind of a scientist engenders new ideas. He sees their fruitfulness and utility, but he knows that they are still so vague that he must go through a long process of analysis to develop them before the public shall be able to understand and appreciate them at their just value. If he believes then that death may suddenly annihilate this whole world of great thoughts, and that perhaps ages may go by before another genius discovers them, we can understand that a sudden desire to live must seize him, and the joy of his work must be confounded with the fear of having to stop it forever.

We can imagine Abel's anguish at the thought of approaching death, when none about him could understand the ideas which he wished to propagate, and which he feared forever lost. We appreciate the moments that Galois must have experienced before fighting his duel, if we remember that a few hours before going on to the ground from which he should not return, he had not written a single line of his great discoveries.

Poincaré died at the most brilliant moment of his career, in full vigor. His spirit was young; original ideas were

[1] A lecture delivered at the inauguration of the Rice Institute, by Senator Vito Volterra, Professor of Mathematical Physics and Celestial Mechanics in the University of Rome. Translated from the French by Professor Griffith Conrad Evans of the Rice Institute.

vibrating in his brain. Did he realize that the world that filled his mind might crumble at any instant, like the wonderful castle of Walhalla, overrun with sudden fire? We cannot say. I hope, for the peace of his last hours, that he did not see death approaching; and yet the pages of his last memoir contain a sad presentiment.

His mind now is resting, and perhaps the present hours are his first hours of repose. In consideration of the activity which he displayed, the number of different questions which he treated, the new conceptions of science which he quickly absorbed, the number of original ideas which he disseminated, we are sure that he could not have rested a moment during his life. Poincaré was ever at the breach, a good soldier, until his death. During these last thirty years there has been no new question, connected even remotely with mathematics, which he did not subject to his deep and delicate analysis, and enrich with some discovery or fruitful point of view.

I believe that no scientist so much as he lived in constant and intimate relation with the scientific world that surrounded him. He received ideas and gave them by a process of exchange both rapid and intense, which ceased only on the day when his heart ceased to beat. That is why, if we were to characterize the recent period of the history of mathematics by a single name, we should all give that of Poincaré, for he has been without doubt the most widely known and celebrated mathematician of recent years. Little by little he created a type of scientist and philosopher. Without being aware of it, the mathematicians of his time, by means of subtle sympathies and bonds, grew necessarily into that type.

Scientific development, the relations of science with life, and of the general public with the scientist have been greatly changed in these late years. The causes are easy to understand, the effects striking. Brilliant discoveries have illumined every department of life. It is for this reason that science in general has become popular, and people expect from the mathematical and physical sciences particularly, results always new and ever more useful. It may be that people even have come to have a confidence in them which surpasses their power. The scientist who a few years ago stayed hidden in his study or in his laboratory to-day mingles with other scientists and with the public. He hears the questions which are asked from every side, and he must reply. Too much urged, he must sometimes reply before his thought is ripe.

Congresses and scientific reunions have increased in number; and popular presentations and learned lectures, where people wish to know the last word of science, follow each other without pause. There is no longer any time to wait. Modern life, eager and tumultuous, has invaded the quiet dwellings of the scientists. Some centuries ago people published great volumes; they were the synthesis of the thought of a man's whole life. But that was not sufficient for the scientific development now in progress. Scientific journals to-day ask for memoirs, in which work is published as it progresses. The proceedings of the academies, short and precise reviews, have appeared. A man reports in a few words every discovery as soon as he has made it. Time presses; one fears that the next minute the discovery may be lost. But the communications of the congresses, which no one has the leisure to revise, exceed in rapidity even the proceedings of the academies and of the scientific societies. We wish to know what has not been done. We say what we hope to find. We confide that which we shall never have courage to print. This development has created a particular state of mind among scientists, and has changed their lives, their ways of working, and even of thinking.

There are great advantages in this modern scientific life as I have just presented it. Research has become almost collective. The energies of the investigators are summed; their discoveries follow each other rapidly; competition spurs them on. Their number increases from day to day. But how many objections we can oppose to these advantages! What refinement of detail is lost! Perhaps that patience, which for Buffon was genius itself, has vanished in the tumult of the present hour. Poincaré was a modern scientist in the full meaning of the word. There was no congress, no scientific reunion where his word was not heard. Most of the scientific journals received his memoirs and the accounts of his investigations. The universities of Europe and America have heard his conferences and lectures.

A work so absorbing, so intense, may easily overdrive to the point of danger a weak or sickly constitution. Is it this excess which fatally has led Poincaré to the tomb?

Calm and serene scientific work is often a rest for the mind. The pleasure of the new results that one finds suddenly, like a beautiful landscape at the turn of a mountain road, alternates with the labor of research. The difficulties of analyzing the question are often generously compensated by the solutions which appear at the precise moment when one expects them least, by means of methods which one could not hope to find useful. The work which Euler, Lagrange, Gauss knew may be compared to a pleasure voyage in the finest of countries; but that which public lectures and conferences demand, which journals ask for at a fixed rate, very often fatigues and irritates like a long and rapid tour, during which one has no time to consider the surrounding beauties and charms.

I imagine that a mind even so largely endowed as that of Poincaré, one that possessed all the gifts appropriate to scientist and author, must have felt fatigue and weariness before a mass of labor which year after year continued without intermission or rest, and every day became more demanding and intense. But modern life called for it, and a famous man like Poincaré, most popular of mathematicians and philosophers, could not refuse.

Perhaps he felt that it was the duty of his genius towards humanity to spread abroad his ideas, not hiding any. He gave as he found, generously, as a great lord who has immense resources and is sure that no hasty expenditure can use them up. He did not hesitate between the desire to make known his thought to a great public and the fear of giving out results not yet completely ripened. An unusual lucidity saved him from mistakes. He always laid bare his ideas, and he did not hide his methods. That ingenious and subtle way of giving results and concealing the manner of getting them, so dear to the ancients and always so tempting, never appealed to him. He never waited to make complete and final his discovery, and give it a systematic and definitive form; although it is exceedingly self-satisfying to stop and investigate from every side that which one has discovered and which is really one's own. It is indeed pleasant to find new aspects of it, and obtain its applications.

But Poincaré resisted all these temptations. He sacrificed these gratifications of the scientist to a high ideal. He went ever ahead. New questions awaited him, and the time for considering the details of the old never came. Indeed, I believe that he consistently avoided details and did not wish to give his time to minute questions. It was not his business either to correct or to revise that which he had done. The whole was everything for him, the details nothing.

This inherent ardor gave to his nervous style a personal stamp and character. Perhaps it is for this reason also that it is impossible to compare Poincaré and other investigators, even those of the present date. He is too modern for any comparison to be possible. Among scientists he is like an impressionist among artists, and I know of no other scientific impressionists among the great men of the past.

It is quite certain that no theory like universal gravitation or electrodynamics will be attached to his name, as to those of Newton, Ampère and Maxwell. Among the great number of methods which he invented and developed from day to day are there any comparable with those which made famous Archimedes or Lagrange? It would take a great deal of time to distinguish everything that there is in his works, in order to say which of the seeds that he has sown will sprout, and which finally will be most fruitful. But if we ask to-day, on the morrow of his death, at what level we should place his genius, we must reply that he has reached the altitudes where dwell the great of human kind. There is certainly a philosophy that is Poincaré's, and an analysis, a mathematical physics and a mechanics that are Poincaré's, which science can never forget.

His renown during his life was great. Few scientists and a very few mathematicians have had celebrity equal to his. A physicist would find the reason for this in what I have just been saying, remarking that his spirit and the spirit of his time vibrated in unison, and that he was in phase with the universal vibration. Some great scientists have labored, urged by an internal force, without hearing or concerning themselves with those about them. They have been misunderstood. The pitch of their voices was not in harmony with that of their times, and they uttered tones which resounded only in later generations.

Nothing is harder than to prophesy the reputation of a scientist. History has given too many contradictions to obvious prophecies. Will not what one wonders at to-day be unessential to-morrow? But it is impossible that Poincaré's voice shall not be heard in future times. The questions that he treated are so important and fundamental that a great number of investigations will follow those which he commenced. His works will be studied in detail, and by many. They will form a very precious mine for all the scientists to come. The wealth of it, even at the present moment, we can surmise.

These last few words explain what I am going to talk about. It is impossible to summarize surely and adequately the entire work of Poincaré, and to give a complete survey of his mind and his wonderful activity. But I wish to devote to him this lecture. His voice should have sounded here in this solemn event, and the Rice Institute should have been inaugurated also under the auspices of his illustrious name. I shall endeavor to recall, then, a very small number of his discoveries, by trying to trace their principal characters, and to show their place in reference to the time when they were developed.

I hope to be excused if I recall facts already known, and if I consider a few details that are elementary. But since I cannot be complete I must be clear, and I shall therefore aim not to describe matters in a difficult manner. I hope that you will understand my selections from his works: I have endeavored to take them from various branches of mathematics in order to show the development of several of his speculations.

I begin with the one of Poincaré's investigations that first brought him to the attention of the mathematical world, and at once showed his great talent in analysis. This is the theory of linear differential equations and of Fuchsian functions.

The theory of functions was the most important conquest of analysis during the last century. I did not hesitate at the Congress of Mathematicians at Paris to call the nineteenth century the century of the theory of functions, as the eighteenth might have been called that of infinitesimal calculus. An intuitive idea, like the idea of function which everybody possesses, and which is related to the most elementary conceptions of quantities which vary with constant laws, gradually has invaded the whole subject of mathematics. Analytic geometry and the infinitesimal calculus gave it a start; algebra gave a great impulse to its systematic study; and Lagrange was able to write the first theory of analytic functions, the celebrated work in which are found the germs of later progress. It is only by the enlargement of the field of variables that the theory has been built up in a precise manner. It was necessary to consider imaginary and complex values in order to be able to explain the most hidden and most important properties of functions. To study a function without considering its imaginary and complex values would in many cases be like wishing to know a book by looking at what is written on the back, without reading the pages that are inside.

Cauchy, Riemann and Weierstrass have assisted us most in the reading of this mysterious book. All of their genius was necessary to lay bare to us its most interesting secrets.

But, as often happens, a general theory can be developed only by means of a profound study of a particular class of the objects which one is considering. Always some guide is necessary to provide orientation in a new region which has not yet been explored. The guide in the theory of functions has been the detailed study of elliptic functions. A great many questions of algebra, of mechanics, of geometry, and of physics lead to the development of this branch of analysis, which has followed so closely that of the trigonometric functions: the elementary functions which Euler had already shown to be related to the logarithms and exponentials.

The history of elliptic functions is well known. It has been written many times, because it is perhaps the most interesting part of the history of mathematics. We pass from surprise to surprise in passing from one step, which we believe to be the most important of its development, to another, which brings forth new discoveries and new surprises. It has happened that the general theory of functions as well as all the other particular branches which are related to it has been cast upon the model of the theory of elliptic functions, and thus it is that the theory of Fuchsian functions, which represents the latest of these constructions, follows it also, in its essential features, according to the plan of Poincaré.

As is well known, the principles upon which the theory of elliptic functions is constructed are three: the theorem of addition, the principle of inversion and that of double periodicity. Everybody has learned in the elements of trigonometry that the sine and cosine of a sum of two arcs can be calculated from the sines and cosines of the arcs themselves, by means of very simple algebraic formulæ. In its specific form the theorem of addition of elliptic functions is quite similar to that which we have spoken of. It is not, however, under this aspect that it first appeared. Fagnano, an Italian investigator who made part of no scientific circle but possessed great talent, recognized it in the geometric properties of a special curve—the lemniscate of Bernoulli.

The genius of Euler was necessary to show the true nature of this property and to develop it in all its generality.

Another most subtle property, made evident only much later, is that of double periodicity. The periodicity of trigonometric functions comes immediately from their very definition. The double periodicity of elliptic functions was not discovered until Abel and Jacobi established the principle of inversion—that is to say, when they had taken the whole theory from the reverse side. Legendre, who thought the theory already complete, had to learn that he had not yet investigated its most fundamental conceptions.

Abel and Jacobi kept on in the route which they had struck out. The general theory of the integrals of algebraic functions was systematically constructed upon the theorem of Abel, which is an extension of the theorem of addition, upon the principle of inversion which Jacobi demonstrated for the first time in complete generality, upon multiple periodicity, and finally, upon the use of certain functions which are called Jacobian functions.

The principle of inversion under a new form, the extension of the idea of periodicity, and a modified type of Jacobian function were carried over at one stroke by Poincaré into the new domain—that of linear differential equations. It was that which constituted his work of analysis upon Fuchsian functions.

After quadratures, the great problem of infinitesimal calculus is the integration of differential equations. The most simple differential equations are the linear ones. We get an equation of this sort if we imagine a relation of the first degree to hold between the displacement of a particle, its velocity, and its acceleration, the coefficients of the equation depending in an arbitrary manner upon the time. The particular equation that we have just defined is of the second order, because the velocity is the first derivative, and the acceleration is the second derivative, of the displacement; but we can imagine linear equations where derivatives appear of any order, and which are accordingly of higher order than the second.

Lagrange and many other mathematicians studied these equations, but Gauss investigated a special class of them completely. He connected them to their series, which was the hypergeometric series. Riemann went still further into these questions. He published a celebrated paper upon the subject; and after his death results of the greatest importance were found among his manuscripts. It seems that Weierstrass, without having published anything, had also discovered much relating thereto. But we owe to Fuchs an article, appearing in 1886, which called the attention of the entire scientific world to the new manner of considering linear differential equations. If we wish to form an idea of the new level to which Fuchs and his predecessors had carried the question, we have only to compare it with the theory of elliptic functions at the time of Legendre—that is to say, before Abel and Jacobi appeared upon the scene.

And yet advances had already been made into the new subject about to be developed, since the theory of the modular function was known.

The integrals of uniform functions are reproduced with the exception of an additive constant when the variable performs a closed circuit round singular points. This property is the origin of the periodicity of elliptic functions. In the same way, the set of fundamental integrals of a linear equation with uniform coefficients is subjected to a linear transformation on going around a singular point. We seek in this remarkable fact the key to the properties of those functions which can be obtained from the linear differential equations, by a procedure analogous to that of the inversion of elliptic integrals.

If the equation is of the second order, the ratio of two fundamental integrals undergoes a linear substitution on performing a closed circuit round a singularity.

We see then that the independent variable regarded as a function of the ratio of the two integrals must remain invariant of certain linear substitutions executed upon this ratio. The property which was to replace that of periodicity was thus found, and at the same time the principle of inversion. Poincaré started from this fundamental idea and interpreted geometrically that which we have just called a linear substitution. He started a systematic study of those substitutions which belong to a single discontinuous group, because it is evident that uniform functions which remain invariant of continuous groups cannot be other than constants.

Linear substitutions correspond geometrically to transformations of the plane by means of inversions by reciprocal radii, united with reflections. They play a very important part in non-Euclidean geometry, as several geometers, among others Beltrami, had already shown. Poincaré distinguishes two kinds of groups, those which he calls the Kleinian groups, which are the most general discontinuous groups, and the Fuchsian groups. These last, interpreted geometrically, leave the real axis fixed; but by composition with a certain new substitution they leave a circle invariant. It is this circle which Poincaré calls the fundamental circle.

The finding of all these discontinuous groups is in this manner reduced to the consideration of the possible regular divisions of the plane and of space. Poincaré distinguished between Fuchsian substitutions of different families, and obtained the corresponding groups. He then had actually to construct the functions which remained invariant of the substitutions of these groups. These are the so-called Fuchsian functions.

Jacobi, starting from elliptic functions, had arrived at a function which he called Θ—that is to say, the Jacobian function. It is not periodic, but possesses what is called periodicity of the third kind, because increasing the variable by one period reproduces the function, multiplied by certain exponentials. Jacobi showed that the simplest way to obtain the theory of elliptic functions was first to define directly this function Θ by means of a series, finding its properties by algebraic methods, and then afterwards to calculate the doubly periodic functions as ratios formed by the Θ functions.

Poincaré followed a similar method for the Fuchsian functions. He started by calculating the Fuchsian Θ functions by means of series, and then found the changes that they underwent by performing upon the variable the linear substitutions of a Fuchsian group. Certain ratios formed by these Fuchsian Θ's remain unchanged when the variable is subjected to substitutions of the same group.

It is thus that the new transcendental functions were invented. By their introduction into mathematics a new field of analysis was created. We shall not enter into the details of the properties of these new functions, upon their connection with algebraic functions, or with Abelian or other transcendental functions. Neither shall we speak of a large number of questions of arithmetic, algebra and analysis which are related to them.

But we must say a word about the relation of the Fuchsian functions with the integrals of linear differential equations that have algebraic coefficients. The direction here taken by Poincaré is similar to the one which we follow when we express Abelian integrals by means of the generalized Θ

functions of Jacobi—that is, by means of the Abelian Θ's. Following this method, Poincaré introduced the Fuchsian Zeta functions, deriving them from the Fuchsian Θ. These are transcendental functions that express the desired integrals.

It has been asked several times, Have the Fuchsian functions applications? But one can answer with the question: What does it mean for a theory to have applications? Does the touchstone of a theory consist in its use in mechanics or physics? Did the theory of conics which the Greeks raised to such a high state of perfection take its honorable place in geometry only upon the day when people believed that those curves were the orbits of planets? Was it not already a great artistic monument, without reference to any practical application?

But we must not spend time upon these matters outside of our subject. Let us now abandon analysis and pass along to other questions.

There are two kinds of mathematical physics. Through ancient habit we regard them as belonging to a single branch and generally teach them in the same courses, but their natures are quite different. In most cases the people who are greatly interested in one despise somewhat the other. The first kind consists in a difficult and subtle analysis connected with physical questions. Its scope is to solve in a complete and exact manner the problems which it presents to us. It endeavors also to demonstrate by rigorous methods statements which are fundamental from mathematical and logical points of view.

I believe I do not err when I say that many physicists look upon this mathematical flora as a collection of parasitic plants grown to the great tree of natural philosophy. But perhaps this disdain is not justified. In the evolution of mathematical physics these researches probably are to play ever an increasing part.

Explain to a child the first propositions of Euclid. It is not the geometric properties which surprise him; rather, that it is necessary to prove them, because his mind is not experienced enough to doubt their obviousness. In the same way, certain theorems which are demonstrated in mathematical physics produce upon some people a similar surprise.

We are not familiar with the development of geometry before Euclid, and we see therefore the complete work. It is quite probable that in the progress of geometry there were periods when feelings similar to those of which we have just been speaking existed, and little by little passed away.

The other kind of mathematical physics has a less analytical character, but forms a subject inseparable from any consideration of phenomena. We could expect no progress in their study without the aid which this brings them. Could any one imagine the electromagnetic theory of light, the experiments of Hertz and wireless telegraphy, without the mathematical analysis of Maxwell, which was responsible for their birth?

Poincaré led in both kinds of mathematical physics. He was an extraordinary analyst, but had also the mind of a physicist. We shall seek for the proof of this among his works.

The memoir that appeared in 1894 in the "Rendiconti di Palermo" is one of his most interesting papers. It bears the title, "On the Equations of Mathematical Physics." The author presents the question which he is about to treat in a short introduction, where he recalls the work of some of those who preceded him. But the question has a long history of which I shall speak somewhat.

Let me begin by saying that the work has a character which is essentially analytic, and that it belongs to the mathematical physics of the first kind. In precisely what then consists the interest of this question, which so many mathematicians have investigated? No physicist would doubt, for example, that an elastic membrane could emit an infinite number of notes, and that there would be an infinite discontinuous scale of them, going from the lowest tone to the highest. The example of sounds produced by an elastic cord or by a rod is sufficient to suggest what ought to happen when one passes from the case of a single dimension to that of two dimensions, and even what ought to result from the consideration of a vibrating body of three dimensions. But for mathematicians it was necessary to give a rigorous proof, and this proof was complicated and hard to find. We must not even suppose that the analytic investigation had the aim of calculating the pitches of the various notes. Any practical application of the calculation was quite far from the thought of the mathematician. It was only the logical point of view which gave importance to the question. Its difficulty increased its attraction and it thus became a question of compelling interest.

Physicists were intuitively aware of the result, not merely on account of the analogy of which I have just spoken, but also from a certain process of induction which has a philosophic value of the highest order, and which can be regarded as the source of several investigations which continued after Poincaré. Lagrange had devoted a chapter of his "Analytic Mechanics" to the theory of small motions. This chapter is one of the finest of his work. The author was able to carry through all the integrations in the case which he was considering, and obtained very simple and interesting formulæ. The periods of vibration of any set of molecules, finite in number, connected among each other by arbitrary restraints, were obtained by Lagrange by means of the roots of an algebraic equation. Now any system can evidently be considered as a collection of molecules arranged in a space of one, two or three dimensions according as we consider a cord, a stretched membrane or a solid body. It is sufficient then to replace the finite number of molecules of Lagrange by these collections which we have mentioned in order to extend his results to the different cases. This is really what is called Lord Rayleigh's principle, and gives a very clear and suggestive point of view in regard to the bearing of the problem. But this principle was not sufficient demonstration for mathematicians.

The question which we have just been considering from the point of view of the theory of sound, is presented also, either in quite the same manner or in similar form, in several other questions of mathematical physics. We meet it when we consider other vibrations which are not acoustical—for instance, those that are electromagnetic. We meet it also in questions of another nature, such as those of the theory of heat.

A single result had been demonstrated rigorously since 1885, in such a way as to satisfy every mathematician. That was the analytic proof of the existence of the fundamental tone—that is to say, the one which corresponds to the absence of nodes and nodal lines in the vibrating membrane. Schwartz had obtained that result when studying certain questions of a different nature. For a long time he had been developing the theory of minimal surfaces—that is to say, the surfaces of equilibrium of a very thin liquid layer in which there is a surface tension (for instance, a layer of water in which soap is dissolved). In the problem of the calculus of variations, to which he was led, it was necessary to distinguish the maxima from the minima. He was thus led to consider the following question: A function of two variables vanishes at the boundary of a region of two dimensions. The ratio of the value of its differential parameter of the second order to its own value is a negative constant at all points of the region. What is the smallest absolute value of this constant? Now the problem of the notes produced by the vibrations of the membrane consists in finding all the values of this ratio. That is why Schwartz's problem is only a particular case of the one we are considering.

The question then was to proceed to calculate all the other values beyond Schwartz's minimum. Already M. Picard had discovered properties of the greatest importance in this direction, and Poincaré had attacked the problem in a work which was published in the American "Journal of Mathematics," but it must be confessed that in this work he was still far from the solution. He took his revenge in the paper which we are about to examine.

We should guess from Lagrange's theorem and Lord Rayleigh's principle that the different pitches ought to appear as the roots of a transcendental function. It was the construction of one of these functions, or, more particularly, the proof of its existence, that Poincaré attempted. Let us see how.

He commences by adding a term to his equation—that is, he considers one that is made up of three terms. The first is the differential parameter of the second order, the second is the unknown function multiplied by a parameter, and the last is a function which he takes as arbitrary. We shall call this equation the auxiliary equation. The primitive equation lacked just this last term. He constructs this arbitrary function by linearly composing n functions by means of certain constant undetermined coefficients. This done, he develops the unknown function, supposed zero on the boundary, in a series of powers of the parameter. This result is reached by the use of Green's functions. He gets in this way an analytic function of the parameter for which the development is valid within a certain circle, and which can be also represented as the ratio of two functions of which the denominator is independent of the variables of integration. By means of processes of extreme subtlety he shows that these undetermined coefficients of which we have just spoken can be chosen so that the two functions shall be entire functions of the parameter. Hence if in the auxiliary equation we replace the unknown function by the ratio of the two functions, giving them this entire form, we see that for all the values of the parameter which make the denominator vanish the auxiliary equation reduces to the primitive equation, and thus it comes about that all the roots of the entire function which appears as the denominator give the values which we were looking for.

Nothing can be simpler than this process which I have been able to summarize in so few words, but it contains a group of thoughts of a marvelous subtlety and fruitfulness.

What I have given is only the first part of Poincaré's memoir. The study of the roots of the functions which resolved the primitive equation, their properties, the developments that they followed, the definite applications to the problems of acoustics and of the theory of heat, give a number of very important results. They have been applied to many similar questions. At present this classic memoir remains as one of the finest monuments constructed by Poincaré; but it is with the methods of integral equations that we now study those problems. Leaving these questions here, however, let us pass on to other problems and investigations.

Some years ago it seemed for a time as if the atomic and corpuscular theories were losing ground. People thought that everything could be explained by means of continuous substances. In mathematical physics partial differential equations were obtained by abandoning entirely the molecular hypothesis. In chemistry also it was heard that the atoms were becoming useless. But a sudden breath dispersed the light clouds which seemed to obscure the corpuscular theories. They are now supreme, and serve to illuminate the various regions of natural philosophy.

Necessarily the old atomic theories continued to advance. Electricity was first recognized as being of corpuscular nature, and little by little in every subject new sorts of atoms appeared. People discovered facts that accorded with the new theories. These theories became even the richest and most fruitful source of new discoveries, and it is for that reason that their reputation has increased from day to day. It has become now so secure that when contradictions are unavoidably presented we do not think of giving up these new ideas, but, rather, have not hesitated to abandon ancient principles whose validity was not doubted enough even to discuss. Little by little the classic theories which seemed set upon eternal foundations have been upset. Even mechanics, which, after Galileo and Newton, came to be regarded as the most secure of all sciences, has been overturned. A new mechanics has been formed, that of relativity. But that perhaps is already to-day an old mechanics. Will there not come from it indeed again an entirely new one, by virtue of the concept of atoms of energy?

Poincaré was associated with the transformation of the old physics and the birth of the new. His criticism and analysis have penetrated modern conceptions from all sides. He was devoted to such questions up to the end of his life, and several of his articles and latest lectures were given to their exposition. And so Poincaré was not only a master in the first kind of mathematical physics, but also in the second.

The electrodynamics of bodies at rest did not present great difficulties after the discoveries of Maxwell and the progress due to Hertz. But that of bodies in motion gave rise to much discussion. Hertz had suggested a special hypothesis in order to pass from the case of rest to that of motion, but experiment proved it to be false, and it is Lorentz's theory which now explains best the latter subject. The celebrated discovery of Zeeman was a great triumph for the conceptions and hypotheses of Lorentz, because these conceptions and hypotheses predicted the doubling of the lines of the spectrum in a magnetic field; and this was the result verified by Zeeman's experiment.

Lorentz's theory was the source of a new order of ideas, including that which I have called the new mechanics. His theory was put in comparison with the principles of mechanics and physics. No contradiction appears with the principles of the conservation of energy or with those of electricity and magnetism. But at the first step a question is suggested to us, namely: Is it possible to determine explicitly the "absolute" motion of bodies, or, rather, their motion relative to the æther, by means of optical or electromagnetic phenomena?

To make the question still more precise: Do optical or electromagnetic phenomena serve to determine the absolute motion of the earth?

If we take account only of the first power of the aberration, the motion of the earth has no influence on any of these phenomena. This negative result has been shown by experiment, and is perfectly explained by Lorentz's theory.

But a celebrated experiment was performed by Michelson and Morley which kept account of the terms depending on the square of the aberration, and even this experiment, as is well known, gave a negative result.

In a famous paper of 1904 Lorentz showed that this result could be explained by introducing the hypothesis that all bodies are subjected to a contraction in the direction of the motion of the earth.

This paper was the point of departure for the later investigations. The results of Poincaré, Einstein and Minkowski followed closely that of Lorentz. In 1905 Poincaré published a summary of his ideas in the "Comptes Rendus" of the French Academy of Sciences. An extended memoir on the same subject appeared shortly afterwards in the "Rendiconti" of Palermo.

The basic idea in this set of investigations is founded upon the principle that no experiment could show any absolute motion of the earth. That is what is called the *Postulate of Relativity*. Lorentz showed that certain transformations, called now by his name, do not change the equations that hold for an electromagnetic medium; two systems, one at rest, the other in motion, are thus the exact images each of the other, in such a way that we can give every system a motion of translation without affecting any of the apparent phenomena.

In Lorentz's theory a spherical electron in motion takes the form of an oblate spheroid, two of its axes remaining constant. Poincaré found the particular force necessary to explain both the contraction of the electron and the constancy of the two axes. This is a constant exterior pressure acting upon the deformable and compressible electron. The work performed by this force is proportional to the variation in volume of the electron. In this way, if inertia and all of the forces are of electromagnetic origin, the postulate of relativity can be rigorously established.

But according to Lorentz all forces, no matter what may be their origin, are affected by his transformation in the same way as the electromagnetic forces. What modifications will it be necessary to introduce into the laws of gravitation, in virtue of this hypothesis?

Poincaré finds that gravitation must be propagated with the velocity of light. We might think, knowing the famous theory of Laplace, that that was in contradiction with astronomical observations. But that is not so; there is a compensating term which removes every contradiction. Poincaré was thus led to propose and resolve the following question: To find a law which satisfies the condition of Lorentz and reduces to Newton's law when the squares of the velocities of the stars are negligible in comparison with the velocity of light.

Those are the fundamental problems and ideas of Poincaré, which have played such an important part in all later researches. The methods employed involve the principle of least action and the theory of groups of transformations, because Poincaré finds that the transformations of Lorentz form a group in Lie's sense. It is enough to have recalled these general ideas. At the present time they are much spoken of. They form the subject of such a great number of scientific papers and popular conferences that everybody knows them and appreciates their importance.

We shall close by speaking of Poincaré's contribution to mechanics. It is the hardest part of his work to analyze. He concerned himself with practically every branch of analytical mechanics: problems of stability, celestial mechanics, hydrodynamics and potential. The problem of the three bodies forms the subject of a great number of his investigations, now become famous, since they aided in revolutionizing classical methods. As is well known, it was Poincaré's memoir on "The Three-body Problem and the Equations of Dynamics" which was crowned with the prize founded in 1889 by King Oscar of Sweden. Important works of Poincaré's followed this memoir: the three volumes entitled "Les méthodes nouvelles de la mécaniques céleste," and the "Leçons" given at the Sorbonne. Moreover, Poincaré's last expository work was devoted to the discussion of the various cosmogonic hypotheses.

The fundamental ideas which guided Poincaré in the problems of mathematical astronomy were the consideration of periodic solutions, the study of the series which give the solution of the problem of three bodies, and the introduction of integral invariants. We have a periodic solution of the problem of three bodies if at the end of a certain time the three bodies are found again in the same relative positions, the whole system being merely turned through a certain angle. By considering the eccentricities and inclinations of the orbits, Poincaré was led to distinguish three kinds of periodic solutions for values of the time infinitely great either negatively or positively.

These studies on periodic solutions have very great theoretical interest, but also they have important practical applications. At a first glance, we can understand that the probability is infinitely small that in any practical problem the initial conditions of the motion will be such as to correspond to a periodic solution. Nevertheless, we can take one of these periodic solutions as a starting-point for a series of successive approximations, and thus study those solutions which differ little from it.

It is well known that a beautiful application of this method was made by Hill to the theory of the moon's motion.

The question of divergence of the series which appear in celestial mechanics has great importance. It is one of the most interesting questions that have arisen in mathematics. Can we use divergent series, and can we by means of series of this kind arrive at approximate solutions of practical problems? The example of Stirling's series allows us to answer in the affirmative. We find series of the same kind in celestial mechanics. They also furnish approximate values sufficient for the demands of practice. That is what Poincaré noticed and proved.

The celebrated theorem about the non-existence of uniform integrals—that is to say, that the three-body problem has no uniform integrals besides those already known—is one of the most remarkable results of Poincaré's theory.

In these researches about which we have been speaking the so-called integral invariants play an essential part. These are approximations which are calculated by quadratures applied to the variables of differential equations, and remain constant. These invariants are connected intimately with the fundamental question of stability.

It is impossible to summarize all these theories and yet present them clearly. On the other hand, to develop them more minutely would carry us too far.

Following the same path that we have taken for analysis and mathematical physics, let us then consider also in mechanics a particular one of Poincaré's investigations, sufficient to show us the range and powerful originality of his genius. On the one hand, this investigation is related to hydrodynamics; and on the other, to celebrated questions of celestial mechanics and, as Sir George Darwin has shown, to the most interesting and modern cosmogonic theories.

It is the question of the equilibrium of a rotating fluid mass, and was one of the first problems that presented themselves with the establishment of the theory of gravitation. MacLaurin gave a solution of it by means of ellipsoids of revolution, and it is perhaps the finest result which that great geometer gave to science. The solution by Jacobi by means of ellipsoids with three unequal axes was a happy stroke of genius of that illustrious mathematician. He was in fact the first to doubt what everybody considered as evident *a priori*—that is, that every possible form of equilibrium of a rotating homogeneous fluid mass is symmetric in regard to the axis of rotation.

But solutions due to MacLaurin and Jacobi were only particular solutions of the general problem. There are an infinite number besides. We must also notice that these particular solutions were not obtained directly. It was merely verified that under certain conditions certain ellipsoids satisfied the laws of equilibrium.

Before considering Poincaré's investigation we must recall the fact that Thomson and Tait in their treatise on natural philosophy had seen that there were ring forms of equilibrium as well as ellipsoids. They had also studied the question of stability, either by imposing certain conditions on the fluid mass—for instance, that of being a solid of revolution or of being ellipsoidal—or by omitting such conditions.

The fruitful idea of Poincaré was that of equilibrium of bifurcation. Let us consider a system whose state depends on a certain parameter. If, for instance, we have a rotating fluid mass, we can let that parameter be the angular velocity of rotation. Let us suppose that several different forms of equilibrium correspond to the same value of the parameter. Let us change that value. The configurations—or, in other words, the forms of equilibrium—will change. It may happen, that, on approaching a certain limit, two forms of equilibrium become the same. If we go by this limit we may have one of two cases. The figures of equilibrium may disappear; we express this in algebraic language by saying that they become imaginary. That is the first case. We say then that that form which the two figures approach is a limiting form. But it may happen that if we pass the limiting value the two distinct figures reappear. That is the second case. In this case the figure where the two forms of equilibrium coincide is called a *form of bifurcation.*

Let us suppose ourselves to be able to represent each figure of equilibrium by a point in the plane of which the coördinates are the value of the parameter and some variable which distinguishes the figure. By changing the parameter we shall have a curve. In our second case this curve is formed of two branches which cross, corresponding at their intersection to the form of bifurcation. Now Poincaré established a theorem of the greatest importance by considering the stability of the figures corresponding to the different points of the two branches. Let O be the value of the parameter which refers to the point of intersection. If for negative values of the parameter there is stability on the first branch and instability on the second, it will be the opposite for positive values of the parameter—that is, there will be instability on the first branch and stability on the other. In other words, there is an exchange of stabilities between the two branches at the place where they cross. This proposition was called by Poincaré the *theorem of the exchange of stabilities*.

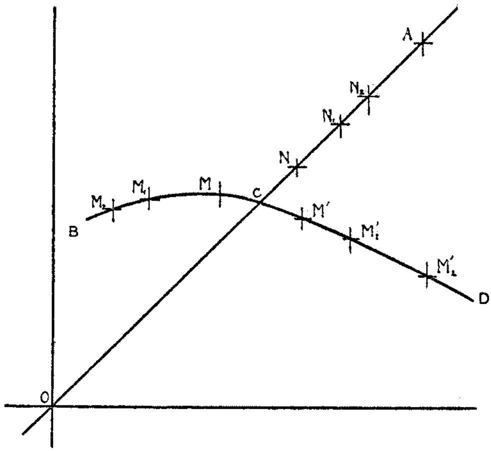

FIGURE 1

Let us now apply these results to the question of the rotation of fluid masses. Let us suppose that we know the solutions of MacLaurin and Jacobi. The axis of rotation is always the small axis of the ellipsoid, and so we know that its ratios to the other axes are less than unity. These ratios are equal for MacLaurin's ellipsoid and different for that of Jacobi. If we take these ratios as coördinates of a point in the plane, each ellipsoid will be characterized by a point, and these points will form a curve (see Fig. 1). The bisector of the angle between the axes will be the line that represents the ellipsoids of MacLaurin. The curve BCD will represent the ellipsoids of Jacobi.

But Poincaré also found new figures of equilibrium that can be obtained by deforming the ellipsoids. The exact form can be calculated by means of Lamé's functions. The simplest have the form of a pear. It is shown that there exist an infinite number of ellipsoids of MacLaurin that correspond to the points N, N_1, N_2 . . . . of the line AO such that one of the infinitely near figures of Poincaré is also a form of equilibrium. In the same way there are an infinite number of points M, M_1, M_2 . . . . M^1, M^1_1, M^1_2 . . . . of the curve BCD such that the neighboring Poincaré figure is also a form of equilibrium.

Let us consider now the stability. The MacLaurin ellipsoids are stable in the part AC, and unstable in the part CO. The ellipsoids of Jacobi are stable from C up to the first point M, where one encounters a figure of Poincaré, and unstable in the part MB.

Hereupon we come to an application of the theory. I quote from Poincaré himself:

"Let us consider a homogeneous fluid initially rotating and cooling slowly. If the cooling is slow enough the internal friction determines that the whole mass revolve with the same angular velocity at all points. The moment of rotation will moreover remain constant.

"At the beginning, since the density is very small, the form of the mass is an ellipsoid which will hold together despite the revolution. The representative point will describe the portion AC of the line which corresponds to the MacLaurin ellipsoids up as far as C, where these ellipsoids become unstable. The representative point, which can no longer take the path CO, will then follow, for instance, the direction CM; the ellipsoid will have its three axes unequal, and this is true as far as M, where the Jacobi ellipsoids become unstable. Beyond this stage, since the mass can no longer keep the ellipsoidal form, that having become unstable, it will take on the only form possible, which is that of the neighboring surface to it. This surface is a piriform figure (see Fig. 2) which has a narrow place in the region marked 3; the regions 2 and 4 tend to increase at the expense of the regions 1 and 3, as if the mass were trying to divide in two unequal parts."

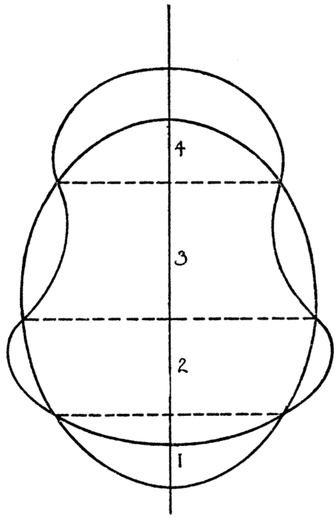

FIGURE 2

The results that we have just presented are quite elegant and of great importance. They revealed much to Sir George Darwin. He thought that the process which we have just described might play a part in the evolution of celestial systems, and this theory seems to be confirmed according to the forms observable in many nebulæ. Some satellites may have been formed in this way at the expense of their planets. In particular that may have happened in the case of the earth and the moon, the masses of which are comparable in magnitude.

Under these majestic aspects, where the most subtle and ingenious theories of mechanics are at one with the most daring cosmogonic hypotheses, we finish our analysis of the investigations due to Poincaré.

I have given but an incomplete idea of the immense work which he did, of the problems which he treated and which it will be necessary to study exhaustively, of the regions which he has opened where several generations of mathematicians will be able to work.

His discoveries will but have the result of stimulating new investigations. That is the fate of the works of great geniuses. They give the key for solving many problems and satisfy scientific curiosity by unveiling the secrets of nature, but at bottom they merely increase that curiosity by opening new horizons and making still more distant the goal of scientific aspiration.

VITO VOLTERRA.